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AN

ELEMENTARY TREATISE

ON

ARITHMETIC,

DESIGNED AS

AN INTRODUCTION

TO

PEIRCE'S COURSE OF PURE MATHEMATICS,

AND AS

A SEQUEL TO THE ARITHMETICS USED IN THE

HIGH SCHOOLS OF NEW ENGLAND.

BY THOMAS HILL.

BOSTON

JAMES MUNROE AND COMPANY

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PREFACE.

It is not the design of this treatise to make arithmetic easy, nor to give such practical rules as may be most useful in business. The Arithmetics in common use are amply sufficient for this purpose.

The present work aims rather at the development of ideas, than the inculcation of rules. It presupposes that the student can cipher, and would only teach him how to think, — how to handle the idea of numbers with which he is familiar, in the same manner in which he must take up, in higher mathematics, the ideas of space, velocity, &c. In this way, it hopes to smooth the path for his future progress in new branches of study.

It presupposes that he can cipher, and will therefore be useless, unless he has previously mastered the treatises of Emerson, Chase, Colburn, or some other writer of school Arithmetics.

The teacher should be careful to make the scholar observe the directions in articles 7, 115, 180, 189, and elsewhere, in respect to practising upon examples. And let him also see that the scholar not only consults the references in parentheses, but understands which part of the article referred to is pertinent to the text.

CAMBRIDGE, January 1, 1845.

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ARITHMETIC.

BOOK I.

PURE ARITHMETIC.

CHAPTER I.

EXPLANATION OF THE LANGUAGE.

1. Arithmetic is the science, the language, and the art of numbers.

As a language, arithmetic is very simple, being capable only of expressing the names of numbers, and four or five simple operations upon them. As an art, it is capable of very extensive uses. (194).*

In strictness it is *an* art, and *a* language; as the science only can be absolute, while the manner of treating and applying it must be variable.

The student has learned to count, that is, to name numbers, and we shall speak simply of the mode of writing them.

---

* Numbers in parentheses refer to articles (as numbered in the margin), and reference to these articles is generally necessary to the understanding of the text.

2. The first nine numbers are represented by the following figures, 1, 2, 3, 4, 5, 6, 7, 8, 9. The figure 0, called zero, is used in notation as an abbreviation of the word nothing.

3. The law of notation by which numbers larger than ten are written is this. Figures are written in a horizontal line, and are increased in value in tenfold progression as you proceed to the left.

Thus, 30 will mean thirty ; 3003, three thousand and three ; 3055000, three million fifty-five thousand ; 3000001008, three billion one thousand and eight.*

Thus, in writing, still more distinctly than in their names, numbers are represented as if made up of a collection of tens, those tens gathered in collections of ten, &c. The right hand place is called the place of units ; the next place to the left, the place of tens, &c.

For other laws of notation, see (87), and Book II., Chap. 11. This is called the *decimal* system.

4. This law would lead us on the right hand of units to tenths, &c., but these have nothing to do with pure numbers.

The law will be extended in Book II., Chap. 7.

---

* We here call a thousand million a billion. This, the French numeration, is better than the English mode of naming a million of millions a billion.

CHAPTER II.

ADDITION.

5. Addition is the finding the name of two or more given numbers, considered as one number. It is expressed by the sign $+$ (called plus), and equality of two or more numbers to others is expressed by the sign $=$.

Thus, $5+7=12$, and $3+4=7$; $3+4+2+1=5+4+1=10$.

6. The sum of every two numbers under ten must be learned, by counting or otherwise, and committed to memory.

Then, by analogy, you can add a number less than ten to any number.

7. To find the sum of several large numbers. Write them under each other, units under units, tens under tens, &c. Add together the numbers whose figures stand in the unit's place, write the unit's figure of the sum under the unit's column, but add the tens to the column of tens. Add the tens, write the units of the sum under the column of tens, and add the tens of the sum to the column of hundreds. Thus proceed to the last column, whose whole sum write down.

It is manifest, that the sum of the whole will then be written according to (3), for the right hand figure will express the number remaining after the tens were separated; the second figure, the tens remaining when the hundreds were separated, &c.

The present article belongs to the art of arithmetic, (1), and of course does not present the only mode of addition.

The only method of gaining a quick hand at this or any operation of mere art, in mathematics, is practice. In the language of mathematics, as in all other languages, familiarity and ease can be obtained only by constantly speaking, reading, and writing. In every chapter of this work, the student must stop and practise the rules here given, upon examples, inventing them for himself, or taking them from treatises previously studied. (See Preface.)

8. In addition, a useful test of one's own work is found in adding up the figures in a different order.

9. Examples.

$1256 + 1783 + 451 + 587 = 4077.$

$15337 + 14001 + 89427 + 98990 = 217755.$

CHAPTER III.

SUBTRACTION.

10. Subtraction is finding the name, as a number, of the difference between two numbers. It is expressed by the sign — (called minus).

Thus, $12 - 5 = 7$, and $13 - 5 = 8$.

11. The larger number is called the minuend; the smaller, the subtrahend; and the difference, the remainder.

12. The remainders, when the minuend is less than nineteen, and the subtrahend less than ten, must be learned, by counting or otherwise, and committed to memory.

Then, by analogy, numbers less than ten may be subtracted from any number.

13. For larger numbers, write the subtrahend under the minuend, units under units, &c. Then, beginning with units, subtract each figure from the one above it, and write the remainder beneath.

This rule is evidently correct, but does not apply to all cases, since a particular figure in the minuend may be smaller than that in the same place of the subtrahend. In this case, it will be necessary to increase the minuend figure by borrowing one from the next higher place, and adding it as ten to the deficient figure; taking care to remember the debt, and consider the next minuend figure as one less than it really stands. If zero should stand in the next place, we must borrow one from the second place. This will be a hundred in the place of the deficient figure, but we need add only ten to it, and put the remaining ninety as a nine in the zero's place. Therefore,

If a figure in the minuend be smaller than the one beneath it in the subtrahend, add ten to it mentally, and consider the next significant figure as one less than it is written; if zeros come between, considering them as nines.

14. Numbers with the sign minus prefixed are called negative numbers. If the negative number be larger than that written as minuend, it must be itself taken as minuend, and the remainder will be negative.

To add negative numbers is to subtract the numbers; to subtract negative numbers is to add the numbers.

15. The test of one's work in subtraction is to add the remainder to the subtrahend, which will produce the minuend.

16. Examples.

1234801 — 678923 = 555878.

$1000002 - 800013 = 199989.$

$1000 - 1005 = -5.$

$1000 - (8 - 5) = 1000 - 3 = 1000 - 8 + 5 = 997.$

$1000 + (5 - 7) = 1000 - 2 = 1000 + 5 - 7 = 998.$

200+6-4 = 200+6

1000
8
992
5
997

CHAPTER IV.

MULTIPLICATION.

17. Multiplication is finding the name, as a number, of a given number repeated a given number of times.

18. The number to be repeated is called the multiplicand, and the number expressing how often it is to be repeated is called the multiplier. The multiplier and multiplicand are called factors. The resulting number is called the product.

19. Either factor may be taken as the multiplier, and the product will be the same.

For each unit in the multiplicand being multiplied by the multiplier, the product will contain the multiplier as often as there are units in the multiplicand.

Hence, in multiplying several numbers, the factors may be taken in any order.

20. The product of every two numbers under ten must be learned, by addition or otherwise, and committed to memory.

21. When the multiplicand is greater, and the multiplier less, than 10, multiply first the right hand figure, then each figure successively, adding to each product the tens of the previous amount, and writing down the units.

22. When the multiplier consists of more than one figure, multiply by each separately; and add together the separate products, placing the right hand figure of each under the figure in the multiplier which produced it.

When there are zeros at the right hand of the multiplier, it is manifest, that they may be neglected during multiplication, and afterwards annexed to the product. The student will need no aid in showing the conformity of all these rules to (3), remembering that he is not bound to show their *exclusive* truth. (1), (7).

23. Multiplication is sometimes indicated by this mark × between the factors, and sometimes by a period between them.

24. A convenient test of one's work is to multiply by parts of the multiplier, and add the products together.

That this sum will equal the product appears from (17).

25. Examples.

$5401 \times 101 = 545501.$

$5431 \times 1020 = 5539620.$

$(54 + 17 + 301) \times (104 - 37) = 24924.$

$16 . 3 = 48 = 7 . 3 + 9 . 3 = 21 + 27.$

CHAPTER V.

POWERS.

26. The product of a factor multiplied by itself one or more times is called a power of that factor.

The factor itself is the first power; its product by itself is the second power. The product of the second power by the first is the third, &c.

27. The number which indicates how often the factor is used in the production of a power is called the exponent.

Any power of a number may be written by writing the exponent above the number on the right hand.

Thus, $5^1 = 5$; $5^2 = 5 . 5 = 25$; $2^4 = 2 . 2 . 2 . 2 = 16$; $3^5 = 3 . 3 . 3 . 3 . 3 = 243$.

28. The exponent of the product of powers of the same number is the sum of the several exponents.

Thus, $5^4 . 5^3 = 625 \times 125 = 5^7 = 78125$. This proposition follows from (26) and (27). $2^3 \times 2^2 = 2 . 2 . 2 \times 2 . 2 = 2^5 = 32$.

29. The exponent of the power of a power is the product of the exponents.

That is, the 3d power of the 5th is the 15th. Which follows from (27). $(2^2)^3 = 2 . 2 \times 2 . 2 \times 2 . 2 = 2^6 = 64$.

CHAPTER VI.

RATIO AND PROPORTION.

30. Ratio is the comparative size of numbers, or the number expressing it.

Even in the mathematical sciences, terms have sometimes more than one signification, (1), and the reasoner in these sciences must cautiously beware of fallacies arising from this ambiguity.

31. Thus we say, 6 is twice as large as 3, or that the ratio of 6 to 3 is 2, meaning that the ratio of 6 to 3 is the same as that of 2 to 1.

But frequently two numbers are not in the ratio of 1 to any number, and a new sign is used to signify ratio.

32. The sign of ratio is a colon between two numbers.

Thus, $5:2$ signifies the ratio of 5 to 2, which is greater than 2 and less than 3. Another method of writing ratio is found in (121).

33. Equality of ratios is called proportion.

Thus, 5, 2, 10, and 4 are in proportion, because $5:2=10:4$.

34. The first term in a ratio is called the antecedent, the second the consequent.

35. In a proportion, the middle terms are called the means, the others the extremes.

Thus in (33) 5 and 10 are the antecedents; 2 and 4 the consequents. Likewise 2 and 10 are the means, 5 and 4 the extremes.

36. If both terms of a ratio be multiplied by the same number, the ratio is evidently unchanged.

37. In every proportion, the product of the means is equal to that of the extremes.

the terms of

For if in any proportion each ratio be multiplied by the consequent of the other, the products by (36) give us the product of the means, and the product of the extremes, bearing the same ratio to the same number, and they will therefore be equal. For instance, $2:3=4:6$, whence $6.2:6.3=3.4:3.6$. But $6.3=3.6$ by (19), whence $6.2=3.4$.

38. Inverse ratio is the same ratio with its terms reversed.

$5:2$ is the inverse ratio of $2:5$.

39. If two products are equal, the multipliers were evidently in the inverse ratio of the multiplicands.

So, too, (37) may be demonstrated by considering, that in the two products the multipliers are in the inverse ratio of the multiplicands, regarding the terms of one ratio as multipliers. (19).

40. Article (37) furnishes a test of proportions by (39). Four numbers are in proportion whenever this test will apply.

That is, 7, 21, 2, 6, are known to be in proportion by the fact, that $2 \times 21 = 7 \times 6$.

41. The difference of the antecedents is to the difference of the consequents as either antecedent is to its consequent.

That is, if equal ratios be subtracted from each other, term by term, the remainders will be in the same ratio. Or that if two numbers be diminished by two numbers having the same ratio as themselves, the remainders will have the same ratio. In which last form its truth becomes evident. But (41) may likewise be demonstrated by (40).

CHAPTER VII.

PRIME AND COMPOSITE NUMBERS.

42. Numbers incapable of separation into equal parts greater than 1 are called prime numbers.

Such are 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, &c.

43. Numbers capable of separation into equal parts may be considered as products. (18), (19). Such are called composite numbers. They are likewise called multiples of their factors.

Thus, 48 is a multiple of the prime (42) factors 2 and 3. And it is a multiple of the composite factors 2^2, 2^3, 2^4, $3 . 2$, $3 . 2^2$, and $3 . 2^3$. As 1 is a prime factor in every number, it is not usually considered at all.

44. Two numbers, which cannot be separated into equal parts of the same size, are said to be prime to each other.

In other words, numbers not multiples of the same number are called prime to each other. Thus, 3, 27, 21, 18, have a common factor, while 3, 5, 8, 77, are prime to each other.

45. When two numbers are prime to each other, if the smaller be subtracted as often as

possible from the larger, and the last remainder be subtracted as often as possible from the smaller number, and so on, each last remainder being subtracted from the previous subtrahend, the final remainder will be 1.

For, if it were greater than one, it is manifest, that the two numbers could be separated into equal parts equal to that final remainder; that is, that the final remainder could be a factor in both numbers, and those numbers would not be prime to each other. (44).

46. If the antecedents in a proportion are prime to each other, the consequents are multiples of the antecedents.

For, if we subtract the ratios from each other, term by term, the remainders will preserve the same ratio, (41), and the final remainder of the antecedents will be 1; (45); if, then, the final remainder of the consequents be zero, the proportion was not true, since it involved the absurdity of two numbers having the ratio 1 : 0; if it be not zero, the numbers were in the ratio of 1 to some number, that is, the numbers were multiple and factor.

47. If the first ratio in a proportion consist of numbers prime to each other, the second ratio consists of multiples of the numbers in the first.

For, if the means be transposed, (40), this case becomes that of (46).

48. Two products cannot be equal, if any factor in one be prime to all the factors in the other.

Which follows from (39, 40, 46, 47), from the fact, that in (46) each consequent must be, by (33), the same multiple of its antecedent; from (19) and (42).

49. Hence, every number is either prime, or the product of certain primes.

That is, when the number 48, for example, is found to be produced by $2^4 . 3$, it is certain that no multiplication of 7 can produce it.

50. Whence the least common multiple of several numbers is formed by the product of all the unlike prime factors, each factor having the highest exponent used in the numbers.

For example, the numbers 6, 8, 14, 18, 27, 24, 15, 12, are equal to $3 . 2$, 2^3, $7 . 2$, $3^2 . 2$, 3^3, $3 . 2^3$, $3 . 5$, $3 . 2^2$, and their least common multiple will be $2^3 . 3^3 . 5 . 7 = 7560$.

51. Multiples of 2 are called even numbers; all other numbers are called odd.

52. Every prime number is either one greater, or one less, than some multiple of 6.

For *a*) it is odd, and, therefore, both one greater and one less than some multiple of two; *b*) every number is either a multiple of 3 (and so not prime), or else one greater or one less than a multiple of 3. Whence every prime number is either one greater or one less than a multiple of 3 and of 2, that is of 6.

53. Every prime number is either one greater or one less than some multiple of 4.

For, if the number one less be a multiple of two by an odd number, that one greater must be a multiple of two by an even number, that is, a multiple of $2^2 = 4$. And *vice versâ.*

Since a prime number is either one greater or one less than a multiple of 6, and one greater or one less than a multiple of 4, the learner may show why the reasoning of (52) would not make it one greater or one less than a multiple of 24.

CHAPTER VIII.

DIVISION.

54. Division is finding the name, as a number, of the parts of a number, separated into a given number of equal parts.

Which, by (18), is equivalent to,

55. Division is finding the multiplicand of a product when the multiplier is given.

Or, by (19),

56. Division is finding one factor of a product when the other is given,

Or part of the prime factors when the rest are given. It will be noticed, that of these three definitions, (54) belongs to the science, (55) to the language, and (56) to the art of arithmetic. (1).

57. Division is evidently a work of successive trials, and approximate results, since it may happen, that the given number is incapable of the required division.

58. The number to be divided is called the dividend, the given factor is called the divisor, and the required factor the quotient.

59. To find the left hand figure of the quotient. Consider how many of the figures at the left of the dividend must be taken to read as a
great number greater than the divisor. From these
or figures so read, subtract the largest multiple of the divisor that you can, and the factor of this multiple will be the required figure of the quotient.

This article is based upon the law of (3), as is also (60).

60. To find the other quotient figures. To the remainder, of art. (59), annex the next figure of the dividend, subtract the largest multiple you can of the divisor, writing the other factor of the multiple as another figure in the quotient. To the remainder, attach the next figure of the dividend, and proceed as before.

61. If the final remainder be zero, the division is complete.

If the final remainder be a number, the division is impossible.

If, after annexing a figure of the dividend to a remainder, it does not read as a number greater than the divisor, zero must be written in the quotient, and another figure be annexed from the dividend.

62. If the divisor be a small number, the process can be performed mentally, writing down merely the divisor, dividend, and quotient.

For instance, to divide 46752 by 3, the work may be written

```
3)46752
  -----
  15584
```

63. But if the divisor be a large number, it will be more convenient to write also the process itself.

Thus, to divide 567523178 by 282, we may write the work thus:

```
282)567523178(2012493
    564
    -----
      352
      282
      ---
       703
       564
       ----
       1391
       1128
       -----
        2637
        2538
        ------
          998
          846
          ---
          152
```

Whence it appears, that the dividend is not divisible by the divisor. To render it so, either 152 must be subtracted, or 150 added.

64. The following remarks can be demonstrated by (3), and the wit of the learner.

a. Every number whose unit's figure is even, is even.

b. Every number whose unit's figure is 5, is a multiple of 5.

c. Every number in whose unit's place is 0, is a multiple of 2×5, and if in the ten's place be also 0, is a multiple of $2^2 . 5^2$.

d. Every number, the sum of whose figures is a multiple of 3 or 9, is a multiple of 3 or 9.

65. When divisor and dividend are manifestly both multiples of the same number, that is, have a common factor, both may be divided by that common factor before proceeding to division.

Since the introduction of a factor into a multiplier will introduce it in the product, (49), in like manner, to remove a factor from a product we need only remove it from the multiplier, without altering the multiplicand. Which demonstrates the present article by aid of (55).

66. The process of (65) is called suppressing the common factors.

67. When all common factors are suppressed, and the division is possible, the divisor becomes 1, and the dividend becomes the quotient.

The prime factor 1, in this case, appears, though usually out of sight. (43).

68. The mark $\div$ may be used to indicate "divided by."

69. Examples. $12 \div 3 = 4$; $183 \div 3 = 61$.
$1764 \div 84 = 294 \div 14 = 21$. See (65), (64, *a*, *d*).
$187110 \div 135 = 4158 \div 3 = 1386$. See (64, *b*, *c*, *d*).
$(100000 - 75076) \div (104 - 37) = 372$.
$100000 - 72800 \div 104 - 37 = 99263$.
$5539620 \div 5431 = 1020$.

70. Division may be indicated by a horizontal line between the dividend and divisor.

Thus, $\frac{12}{3} = 4$; $\frac{24924}{372} = 67$.

This mark of division may be used without regard to the possibility of the division. Thus, $\frac{3}{4}$ may be read, three divided by four, although such division (in pure numbers) is impossible.

71. A convenient test of our work, in division, is multiplying the quotient by the divisor. And, in multiplication, dividing the product by one factor.

CHAPTER IX.

NEGATIVE AND DIVIDED EXPONENTS.

72. When a higher power of a number is divided by a lower power, the quotient is a power whose exponent equals the exponent of the dividend minus the exponent of the divisor. (27).

73. Hence, any divisor may be written as a factor (23) with the sign minus before its exponent.

74. By (49) any number may be written in separate prime factors, each with its requisite exponent.

For instance, 48, 49, 50, 2058, may be written $2^4 . 3$, 7^2, $2 . 5^2$, $2 . 3 . 7^3$.

75. By (23, 73, 74), division may be performed as a sort of negative multiplication. Write the divisor and dividend by (74, 73), and multiply them by (28, 23).

For instance,

$$2058 \div 98 = \frac{2 . 3 . 7^3}{2 . 7^2} = 2^0 . 3 . 7 = 21.$$

$$\tfrac{144}{30} = 3^2 . 2^4 \times 5^{-1} 3^{-1} 2^{-1} = 5^{-1} 3^1 2^3 = \tfrac{24}{5}.$$

It will be noticed that this process is the same as that of (65, 67), differing only in the language. For the form $\frac{24}{5}$, see (70).

76. When a number is divided into equal factors, those factors are called roots of the number, and are called first, second, third, &c. roots, as the number of factors is one, two, three, &c. From (26) it follows that power and root are reciprocal terms. And from (27, 29), that the exponent of the root is the exponent of the power divided by the order of the root.

Thus, the third root of 216 is $216^{\frac{1}{3}} = (3 . 2)^{\frac{3}{3}} = 3 . 2 = 6$.

And the fourth root of 256 is $256^{\frac{1}{4}} = 2^{\frac{8}{4}} = 2^2 = 4$.

77. In these examples, the upper exponent shows how often the number is to be taken as a factor, and the lower exponent shows into how many equal factors that product is to be divided. Such compound exponents may sometimes indicate impossible operations. As, for instance, $2^{\frac{3}{2}}$ signifies the second root of 8, yet no two equal factors can produce 8.

78. To devise a method of finding one of several equal factors in a number, that is, of extracting any root of a number.

This, like division, may, in some cases, (as mentioned in (77),) be an impossible thing.

A. Every power of 1 is 1, and 10 raised to the same power is written as 1 with zeros annexed in number equal to the exponent of the power, and 100 raised to the same power will have twice as many zeros, and so on. Hence, (3), if we separate a given written number into periods containing as many places as the exponent of a required root, the number of periods will show of how many places of figures the root consists.

B. The first (left hand) figure of the root is the largest root contained in the first period. For by (3) the subsequent figures cannot equal an additional unit in the first left hand place.

C. Each figure of the root will be multiplied by the figures on its left raised to a power one less than the given power, and by the exponent of the power. This product will be the largest multiple of that figure contained in the given number.

If the induction which leads to (C) be beyond the grasp of the learner, let him illustrate it to himself in a particular example. Thus we may raise 57 to the third power, keeping the products of each figure separate, but giving them their proper decimal (3) places.

$$\begin{array}{cccc}
 & & 5 & 7 \\
 & & 5 & 7 \\
\cline{3-4}
 & & 7.5 & 7^2 \\
 & 5^2 & 7.5 & \\
\cline{2-4}
 & 5^2 & 2(7.5) & 7^2 \\
 & & 5 & 7 \\
\cline{2-4}
 & 7.5^2 & 2(7^2.5) & 7^3 \\
 & 2(7.5^2) & 7^2.5 & \\
5^3 & & & \\
\hline
5^3 & 3.7.5^2 & 3(7^2 5) & 7^3
\end{array}$$

Here, the multiple of 7, which is highest in decimal position, and which is consequently the largest, is, according to (C), a multiple of 7 by the [3 — 1] second power of 5 and by 3.

Hence we may try whether a number be composed of a given number of equal factors, that is, attempt to extract any root of a number, as follows:

a. Separate the number into periods according to (A).

b. From the left hand period subtract the largest power possible (whose exponent equals the exponent of the required root), and its root will be the left hand figure of the required root, by (B).

c. The second figure is found, according to (C), by dividing the remainder by the first figure raised to a power whose exponent is one less than that of the required root, and multiplied by the exponent of the required root.

d. By similar reasoning, we may now from the two left hand periods subtract the two figures of the root raised to the required power, and obtain a third figure from dividing the remainder by the two figures raised to the next lower power, &c.

79. In Book II., Chap. 8, the learner will find convenient rules for discovering the second and third roots of numbers, after studying which, he may return to article (78).

BOOK II.

ARITHMETIC IN ITS APPLICATION TO QUANTITY.

CHAPTER I.

PRELIMINARY REMARKS.

80. In pure arithmetic, we endeavoured to use chiefly demonstrative reasoning. That is, we reasoned from principles which the mind, by a rapid survey of many examples, perceives to be true in all cases.

For instance, such principles as (36), and those stated in (52, *a*) and (52, *b*).

81. In arithmetic applied to quantity, (mixed arithmetic,) we more freely use inductive reasoning. That is, we shall reason from principles which the mind, by a slower survey of examples, perceives to be true, as far as it can see, in all cases.

Such a principle, for instance, as that developed in (78, C).

82. Numbers, being in themselves abstract and indivisible, are most readily applied to those separate and individual things which first suggested the ideas of them to the mind.

Such things are men, animals, stones, &c., in the visible world, and figures, points, moments, eras, &c., in time and space.

83. But numbers can also be applied to continuous quantity.

Such as extension, change of direction, duration, &c., in space and time, and surfaces, solids, velocities, &c., in the outward world.

84. This is done by selecting a certain quantity as the unit of that kind of quantity, and if a given quantity consists of a certain number of times this unit, it may be represented by this number.

85. The laws of abstract number being simple and uniform, the solution of a question of quantity is made easier by writing the quantities in numbers. After treating these numbers by the laws of pure arithmetic, the result is to be interpreted according to the conditions of the problems.

Thus, 5 bushels and 80 cents being given, the product 400 may represent how many cents five bushels cost, or how many bushels 80 cents will buy. And the quotient 16 may show the price, in cents, of a bushel, or how many parcels of five bushels each 80 cents will buy.

86. The decimal law of notation (3) is also used in mixed arithmetic, by which there is, as it were, a series of units formed, each being ten times the size of the next smaller.

This is most distinctly seen in the currency of our country, in which the cent, the dime, the dollar, and the eagle, are each recognized as really a distinct unit. In France, all weights and measures conform to a decimal scale.

87. But, in very many kinds of quantity, other laws of notation are combined with this, and the different units proceed by an irregular progression, though always written increasing from right to left.

88. Some of the most familiar instances are here given.

a. In English currency,

4 farthings make 1 penny.
12 pence " 1 shilling.
20 shillings " 1 pound.

To indicate that figures represent values in this currency, the letters *d.*, *s.*, and £. are annexed to them, signifying penny, shilling, and pound.

b. In measuring change of direction in a plane, or angular measure,

60 seconds, marked ″, make 1 minute, marked ′.
60 minutes " 1 degree, " °.
360 degrees make a revolution, or return to the same direction.

c. In measuring duration or time,

60 seconds, marked s.,	make	1 minute,	marked	m.
60 minutes	"	1 hour,	"	h.
24 hours	"	1 day.		

A day has two absolute values. The sideral day is the time in which the earth revolves upon its axis. The solar day is variable in its length, but its mean value is longer than the sideral day nearly in the ratio of 21659 to 21600.

d. In measuring extension in one dimension, and lengths,

3 barleycorns	make	1 inch,	marked	in.
12 inches	"	1 foot,	"	ft.
3 feet	"	1 yard,	"	yd.
11 yards	"	2 rods,	"	rd.
40 rods	"	1 furlong,	"	fur.
8 furlongs	"	1 mile.		

The absolute value of a foot is not taken like that of measures of time from any natural standard. But we can refer it to the standard of time, and thus fix its value. According to Captain Kater, a pendulum at the level of the sea, in a vacuum, at latitude 51° 31′ 8″, which beat mean solar seconds, measured in length 39 inches and 13842 of the hundred thousandths of an inch.

The foot also might be compared with the second by means of longitude. For two places east and west of each other, at one of which every star came to the meridian one sideral second later than that at the other, must be at a fixed distance from each other, depending on their latitude. At Cambridge, their distance would be about 1077 feet.

But in France they have derived their measures of length, and all their measures of size, from a natural standard, by actual measurement of the size of the earth, and by adopting a certain part of that size as their unit.

e. In measuring extension of two dimensions and surfaces,

144 square inches make 1 square foot.
9 square feet " 1 square yard.
121 square yards " 4 square rods.
40 square rods " 1 rood.
4 roods " 1 acre.
640 acres " 1 square mile.

A square inch is a square whose side is an inch in length.

f. In measuring extension of three dimensions and solids,

1728 cubic inches make 1 cubic foot.
27 cubic feet " 1 cubic yard, &c.

A cubic inch is a cube whose face is a square inch.

g. In measures of space occupied by dry substances, in ordinary trade,

8 quarts make 1 peck.
4 pecks " 1 bushel.

A bushel is 2150 cubic inches and 42 hundredths of a cubic inch, and is measured in a cylinder 8 inches deep, and 18 inches and a half in diameter.

h. In measures of space filled by liquids, in ordinary trade,

4 gills make 1 pint.
2 pints " 1 quart.
4 quarts " 1 gallon.

A gallon is 231 cubic inches.

i. In measuring weight of ordinary substances, or avoirdupois weight,

16 ounces, marked oz., make 1 pound, marked lb.

112 pounds make 1 gross hundred weight, marked cwt.

20 hundred weight make 1 ton.

A cubic foot of pure water, at 56° Fahrenheit, weighs 1000 ounces avoirdupois.

j. In the weight of precious metals, or Troy weight,

24 grains make 1 pennyweight, marked dwt.
20 pennyweights make 1 ounce.
12 ounces make 1 pound.

A pound avoirdupois weighs 7000 grains Troy.

k. In the mixing of medicines, or apothecaries' weight,

20 grains make 1 scruple, marked ℈.
3 scruples " 1 drachm, " ʒ.
8 drachms " 1 ounce, " ℥.
12 ounces " 1 pound, " ℔.

The ounce Troy and that in apothecaries' weight are equal.

89. From (88, *d*), it appears that measures of time and space, which are entirely heterogeneous, are connected by means of force, a species of quantity which is not homogeneous with either space or time, but which connects them, because it is manifested in both at once.

For it is the force of gravity whose intensity determines the time of the pendulum's swing, and the earth's rotation.

From (88, *i*, *j*, *k*), it appears, that force itself is *quantitative*, that is, susceptible of measurement; and likewise from (88, *a*), that value comes thus by measurement under the power of arithmetic.

CHAPTER II.

ADDITION.

90. Addition, in mixed arithmetic, is finding a number, which bears the same ratio to 1, that the sum of several quantities does to their unit. (84).

That is, it is the expression, in one number, of the sum of several quantities, which are expressed in numbers.

91. This is done by adding the numbers, (5), which express the several quantities.

92. When the quantities are expressed in purely decimal notation (3), the rule of (7) is still applicable.

93. If they are expressed in the mixed notation of (87, 88), write the numbers of each unit under those of the same size. Attempt the division of the sum of the smallest units by that number which makes one of the next greater. Write down the final remainder as the number of smallest units in your required number or sum. Add the quotient to the sum of the next larger units, which then divide as before.

For instance, it is now October 31, 1844, 43 minutes 22 seconds past 9 o'clock, A. M. What will be the date in 17 years, 3 months, 5 days, 2 minutes, and 47 seconds?

yrs.	m.	d.	h.	m.	s.
1843	9	30	9	43	22
17	3	5	0	2	47
1861	1	4	9	46	9

It will be 46 minutes 9 seconds after 9 on the morning of February 5, 1862.

a. The reason why we begin with the addition of seconds is obvious, since the addition of minutes, hours, &c., will never leave odd seconds to disturb the result, whereas, if we begin by adding the years, the odd months and days may make up a year to change the result.

b. 35 days is the sum of the days. Since the three months carries us into February, we subtract 31 for January, which leaves four whole days in February, and it will be the fifth day.

What will three parcels of silver weigh, one containing 11oz. 15dwt. 23gr., another 10oz. 17dwt. 21gr., and the third 9oz. 19dwt. 18gr.?

lb.	oz.	dwt.	gr.
	11	15	23
	10	17	21
	9	19	18
2	6	13	14

c. We begin with grains for the reason mentioned in *a.* The sum is 62, which, divided by 24, gives a quotient 2 and remainder 14. The sum of the pennyweights is 51, and the 2 being added from the grains, give us 53, in which there are 2oz. and 13dwt. remaining. The sum of the ounces is 30, and the two from the pennyweights being added, gives 32, in which are 2lb. and 6oz. remaining.

94. The identity of the operations in (7, 92, 93), should be noticed. The addition of numbers decimally written begins with units, and proceeds up through tens, hundreds, &c., as the addition of numbers written in the system of (87) begins with the lowest units and proceeds to the higher.

And does it for the reason mentioned in (93, *a*), and in a manner analogous to (93, *c*). The addition of numbers decimally written is illustrated by a man counting a heap of cents. He gathers the cents into piles of ten each, then sets those piles in rows of ten, and puts ten rows by themselves. He then easily writes down the sum, writing first the number of sets of rows, then the odd rows, then odd cents, and reads them as so many dollars and cents. This is the addition of units. If he has also in his heap silver coinage, he counts a half dime as 5 cents, or five in his place of units; a dime as equal to a pile of cents, or one in his place of tens; a half-dollar as equal to five piles, or five in his place of tens; a dollar as equal to one row, or one in his place of hundreds. Gold coinage will be, a half-eagle equal to five rows, or five in the place of hundreds; an eagle equal to one set of rows, or one in his place of thousands.

CHAPTER III.

SUBTRACTION.

95. Subtraction, in mixed arithmetic, is finding a number which bears the same ratio to 1, that the difference of two quantities does to their unit.

Or, in other words, it is finding the expression in one number, for the difference of two quantities expressed by numbers.

a. If the quantities be expressed in the decimal notation simply, the difference of their numbers, obtained by (13), will express their difference.

b. If they be expressed in the mixed notation of (87), write the subtrahend under the minuend, (11), each kind of units under its own kind, and, beginning at the right hand, subtract each from each.

96. When, in any kind of units, the minuend figure is the smaller, proceed in a manner analogous to that of (13), by borrowing one from the next larger kind of units in the minuend, and if vacant places come between, considering them as filled by the surplus of your loan.

97. The reason for these proceedings appears from similar considerations to those in (93, *a*). We begin with subtraction at the right hand, lest, if we began at the larger units, the exigence of (96) should force us to change the result after we had written it.

For instance, I have in a barrel 13 gallons, 3 quarts, and a gill of cider, and if I let 4 gallons and three half-pints run on the ground, what will be left in the barrel?

gal.	qt.	pt.	gi.
13	3	0	1
4	0	1	2
9	2	0	3

Since 2 gills cannot be subtracted from 1, we borrow one quart, and mentally consider the minuend as reading 13gal. 2qt. 1pt. 5gi.

If we began at gallons, we should, after writing 9gal. 3qt. in the remainder, be obliged to come back and take one of the 3 quarts to take our one pint from.

98. The identity of the operations in (13) and (97) should be noticed.

99. Examples.

11℥. 3ʒ. 2℈. 19gr. — 9℥. 5ʒ. 1℈. 17gr. = 1℥. 6ʒ. 1℈. 2gr.

£15. 7s. 6d. — 7d. = £15. 6s. 11d.

£18. 9d. — 10d. = £17. 19s. 11d.

CHAPTER IV.

MULTIPLICATION.

100. Multiplication, in mixed arithmetic, is finding a number which expresses a quantity bearing the same ratio to that expressed by the multiplicand, that the quantity expressed by the multiplier does to its unit. (17, 18, 84).

Similar quantities are added together, and subtracted from each other, but quantity cannot be multiplied or divided by quantity. For instance, one heap of mud can be added to another, but it is absurd to talk of multiplying one heap of mud by another. It is plain, then, that

101. The multiplication of one quantity by another can only mean the multiplication of one quantity by the number expressing another quantity, or the increase of a quantity in the ratio of another quantity to its unit.

Exceptions to (101) will be found in (163, 170).

102. But, from (85, 19), the numbers expressing the two quantities may be multiplied, using either of them as the multiplier, and the product will express a quantity of the same kind with that factor that is logically the multiplicand.

For instance, 115 pounds of oakum at 7 cents a pound, will cost 805 cents. Here 7 cents is logically the multiplicand, since the cost of the whole will be a certain number of times the cost of a pound. Yet, practically, 7 is taken as the multiplier, because it is the smallest number. The student will find it a good exercise to distinguish always between what belongs to arithmetic as a science, and what belongs to it as an art and a language. (1, 56, &c.).

103. When the multiplier is expressed in the notation of (87), the question is usually an absurdity. If it be not, the multiplier, or multiplicand, or both, must be changed in their expression, so that the multiplier may be numbered by (84) and (3), before (100, 101) can be fulfilled.

For instance, what is the cost of 3lb. 5oz. 16dwt. of silver, at 5s. 3d. an ounce. Here is the multiplicand given, 63d. The multiplier must be reduced to that unit (84, 100) to which the multiplicand corresponds, namely, ounces. Now, 41oz. at 63d. will amount to 2583d. But the 16dwt. cannot be reduced to ounces. It bears the ratio $16 : 20 = 4 : 5$ to an ounce. Then, by (100), the cost of 16dwt. must be to 63d. in the ratio $4 : 5$. That is, by (33) and (37), four times 63d. will equal five times the cost of 16dwt. Hence, 16dwt. will cost 252d. divided by 5, or 50d. and (70) the impossible quotient $\frac{2}{5}$. So the whole cost of the silver will be between 2633 and 2634d. That is, 3lb. 5oz. 16dwt. of silver at 5s. 3d. the ounce will cost part of a penny more than £10. 19s. 5d.

Practical and easy rules for solving the cases which are included in (103) can readily be formed by the student after he has studied Chap. 6.

104. Exceptions to (103) will be noticed in (172).

105. When the multiplicand only is expressed in a mixed notation, (87), multiply first the smallest units; attempt the division of the product, (18), by that number which it takes of these units to make one of the next larger; write down the final remainder, and add the quotient to the product of the next sized units, &c.

Since multiplication is in its results only an addition of a quantity several times to itself, (17), the reasoning and analogies of (93, *a*) and (94) will sufficiently explain (105).

What is the weight of 5 boxes of sugar, each weighing 3cwt. 1qr. 23lb.

Here $(5 \times 23) \div 28 = 4$, and 3 remain. Again, $(5 \times 1 + 4) \div 4 = 2$, and 1 remains. Lastly, $5 \times 3 + 2 = 17$. So that the gross weight of the 5 boxes is 17cwt. 1qr. 3lb.

106. When the multiplier in (105) is a large number, it is sometimes convenient to separate it, by subtracting such a number that the remainder may be the product of small factors. Multiply by these successively, and to the final product add the product of the subtracted portion.

For a justification of this rule, compare (85, 17, 19).

For instance, a clock loses 1 minute and 33 seconds a day; how much too slow will it be in 53 days after it is set to true time?

Now, $53 = 6 . 8 + 5$, and $6 \times$ 1m. 33s. = 9m. 18s. Again, $8 \times$ 9m. 18s. = 1h. 14m. 24s., to which add

5 × 1m. 33s. = 7m. 45s., and you have the whole loss, 1h. 22m. 9s.

In like manner other numbers may be divided. For instance, 97 = 8 × 12 + 1; 117 = 10 × 11 + 7; 277 = 11 × 25 + 2, and the like.

CHAPTER V.

DIVISION.

107. Division being the reverse of multiplication, (55), the quotient in mixed arithmetic properly represents a quantity, to which that expressed by the dividend bears the same ratio as that expressed by the divisor bears to its unit. (100).

108. But, as in (102), the factors may change places, and yet only one is logically the multiplicand, so here,

109. The divisor given may be logically the quotient.

110. Hence, there are two sorts of problems in division; it may seek to divide a given quantity into a given number of parts, and find the number expressing one of those parts; or, it may seek how often one quantity can be contained in another.

In either case the operation is the same, the quotient in one case being the logical quotient, expressing a quantity; and, in the other, being logically the divisor, expressing a number.

Thus, whether I ask how many men can receive 5 dollars each from a purse of $25, or how many dollars apiece five men can take from the purse, I in either case alike divide 25 by 5. Yet, in one case, the question really is, how many *times* five dollars in twenty-five, and the answer, five times, gives me logically the multiplier, by which I must multiply five dollars to make $25; and, in the other case, the question really is, How many *dollars* in the fifth part of $25, and the answer is logically the multiplicand, which must be multiplied by five to make $25.

111. If both quantities are expressed in decimal notation, (3), the rules of Book I., Chap. 8, are to be followed.

112. If one or both quantities are expressed in the mixed notation of (87), there arises one of three cases (113, 114, 115).

113. If the given divisor be the logical divisor, and be expressed in decimal notation; divide the largest units first, write down the quotient, and to the next smaller units add the remainder expressed in those smaller units; this sum divide as before, &c.

It is manifest that we must begin with the largest units, in order that the quotient may be secure from change when partly written. (93, *a*; 97, 105).

15 men would divide £16. 11s. 3d. among them, what is the share of each? $\frac{16}{15}$ = £1 and £1 left undivided. Add this as 20s. to the 11s. Then $\frac{31}{15}$ = 2s. and 1s. left undivided; add this as 12d. to the 3d. Then $\frac{15}{15}$ = 1d., and each man's share is £1. 2s. 1d.

114. If the given divisor be the logical divisor, but expressed in the mixed notation of (87); reduce it to decimal notation, by multiplying the largest units by that number which one of them contains of the next smaller, then adding the smaller, &c. Then proceed as in (113).

If the divisor contain smaller units than that in reference to which the problem is put, (107), the quotient must be multiplied after it is obtained.

For instance, 3lb. 6oz. of quicksilver cost me £1. 2s. 6d.; what is that a pound?

Here are 54oz. Now, $54 = 6 \times 9$; see (107, 106); and £1. 2s. 6d. divided by 6 = 3s. 9d. This divided by 9 gives 5d. an ounce. But as the price is asked for a pound, we multiply 5d. by 16, which gives 80d. = 6s. 8d.

115. If the given divisor be logically the quotient, reduce both quantities to numbers based on the same unit, (84), and proceed as in (113).

The necessity of this proceeding is self-evident. Unless, indeed, by (105, 17, 55, 107, &c.), you subtract the given divisor repeatedly from the dividend, and count the number of subtractions. Or, subtract multiples of the given divisor, and then add together the different factors of those multiples.

From a lump of blue mass weighing 1℥. 2ʒ. 1℈. 5gr., how many boxes of pills can I make, 1℈. 5grs. in a box?

Here, the whole mass weighs 625grs., and each box contains 25grs. $625 \div 25 = 25$ boxes. Or we may say

$$\begin{array}{rcccc}
 & \text{℥.} & \text{ʒ.} & \text{℈.} & \text{gr.} \\
 & 1 & 2 & 1 & 5 \\
\text{1℈. 5gr.} \times 20 = & 1 & 0 & 1 & 0 \\ \cline{2-5}
 & & 2 & 0 & 5 \\
\text{1℈. 5gr.} \times 5 = & & 2 & 0 & 5 \\ \cline{3-5}
 & & & & 0
\end{array}$$

$20 + 5 = 25$ boxes.

The learner, in practising upon examples in treatises previously studied, (as he must do in all parts of this work,) should point out to which case, (112), each example belongs.

CHAPTER VI.

VULGAR FRACTIONS.

116. It may happen that a given quantity is not a multiple of the quantity assumed as a unit. It cannot then be expressed in numbers, and the part smaller than the unit, the fragment or fraction of a unit, by which the given quantity exceeds a multiple of the unit, must be expressed in some other way.

117. One method is immediately suggested by

the law of (3). If we annex figures to the right hand of the unit's place, making some convenient mark of separation, and suppose them to decrease in tenfold progression, it is manifest, that the decimal fractions thus written will enable us to express parts of the unit with great exactness. Of this we shall speak particularly in Chap. 7. Yet other methods of writing them are sometimes preferable. Suppose, for instance, we wish to express 200 minutes as hours and parts of an hour. There will be 3 hours and twenty minutes. But, as 6 minutes are one tenth of an hour, twenty minutes will contain 3 tenths of an hour and two minutes.

Now, as we cannot divide an hour into hundredths, to find how many hundredths there are in two minutes, we must multiply 2 by 100, and then divide by the minutes in an hour. (For the quotient will be the same if any divisor of the divisor be made a multiplier of the dividend.) And, again, we must multiply the remainder by 10, and divide by 60 to obtain the thousandths, and so on. But, since 10, 100, 1000, &c., and the remainder 2, do not contain the prime factor 3, which is contained in 60, it is plain, that there will be no division possible, and the line of decimal fractions will be continued for ever. Putting a comma for a decimal point, to indicate the separation between the hours and the parts of an hour, we should have 3,33333 &c., for ever. Hence, we must devise some other plan of writing it.

118. Fractions usually result from divisions in which the dividend and divisor contain factors prime to each other, and they are usually written as unexecuted divisions, in the manner of (70).

For example, two hundred minutes divided by 60, gives us $3+\frac{20}{60}$, but $\frac{20}{60}$ is equal, by (66) or (75), to $\frac{1}{3}$, which may now be read one third, and the two hundred minutes, or three and one third hours, may be written $3\frac{1}{3}$h.

119. Fractions thus written are called vulgar fractions. The dividend number is called the numerator; the divisor the denominator. The numerator and denominator are called the terms of the fraction.

The denominator *names* the size of the parts compared with the unit; the numerator *numbers* the parts in the fraction.

120. Fractions may then be regarded as unexecuted multiplications and divisions, or as quotients possible or impossible. (70).

For, as divisors are but factors, (73), and factors may be taken in any order, (19), the unit may be multiplied by the numerator before being divided by the denominator. That is to say, the fourth part of three bushels is three fourths of one bushel, and $\frac{2\times 1}{3}=2\times\frac{1}{3}$.

121. Fractions represent the ratio of the numerator to the denominator.

For, as the ratio of 6 to 3 is 2, that is $=2:1$, so the ratio of 3 to 6 is $\frac{1}{2}$, that is $=\frac{1}{2}:1$.

122. Hence, the form $\frac{3}{4}$ may have three meanings. Either $3:4$, or $3\div 4$, or three fourths. But, in either case, the operations performed are the same, namely, 3 is used as a multiplier, and 4 as a divider.

123. It may chance that a given quantity is not only not a multiple of the unit quantity, but that no multiplication can make it so, in which case it can of course be represented by no fraction. In other words, the ratio of two quantities is sometimes incapable of expression by figures, in which case the two quantities are said to be incommensurable.

Such quantities are analogous to numbers which are prime to each other. (44).

124. An approximation to the value of such ratios may be pushed to any extent by a series of fractions whose value decreases constantly.

For instance, the ratio of the diagonal to the side of the square cannot be fully expressed in figures. It may be attempted in vulgar fractions, thus: $1+\frac{1}{3}+\frac{1}{13}+\frac{1}{256}+$ &c. Or thus, $1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16}-\frac{5}{128}+$ &c. Or, in the decimal fractions of (117), 1,41421 &c.

125. When several fractions have the same denominator, it is called a common denominator.

126. Before performing any other operation on a vulgar fraction, it is generally best to strike out the factors common to both terms. The product of *all* these ~~terms~~ is called the greatest common fact measure.

127. To find the greatest common measure of two numbers.

The remainder in division must contain all factors common to divisor and dividend, and a remainder, which measures the divisor, must also measure the dividend.

Divide the greater number by the smaller, the smaller by the remainder, the first remainder by the remainder of the second division, &c.

The last divisor used will be the greatest common measure.

For instance, what is the greatest common measure of 54 and 258?

54)258(4
 216
 42)54(1
 42
 12)42(3
 36
 6)12(2
 12
 0

Since 54 is a multiple of the greatest common measure, 216 is also, and since 258 is, 258 — 216 must be also. And so must 54 — 42 be a multiple of the greatest common measure, being the difference of two such multiples. In like manner, we show 6 to be such a multiple. The last divisor, 6, contains, then, the greatest common measure. That it contains no other factors is evident, since it measures 12, and must, therefore, measure $6 + 3 \times 12 = 42$, and also $42 + 12 = 54$, and $42 + 4 \times 54 = 258$. What we have thus shown of 6 in this instance is evidently true of the last divisor in every case, since "the remainder in division must contain all the factors common to dividend and divisor, and any remainder which measures a divisor, measures also the dividend."

When the greatest common measure is 1, the numbers are said to have no common measure, agreeably to a remark in (43).

128. If both terms of a fraction are divided by their greatest common measure, the fraction will be put in its lowest terms.

129. To reduce fractions to others of equal value, having a common denominator. (125).

The common denominator cannot be prime to either of the given denominators by (122, 121, 46, 40). But it may be the least common multiple, (43, 50), for any multiplication of the numerator is always possible. Hence

a. The common denominator may be the least common multiple of the given denominators.

b. The new numerators may be formed by multiplying each numerator, (36), by those factors in the common denominator which are prime to its denominator.

130. Whole numbers may be regarded as fractions whose denominator is 1.

For a fraction represents a quantity which bears the same ratio to 1 that the numerator does to the denominator. (121, 84).

It represents also an unexecuted division and multiplication. (120). So a whole number represents the multiplication of the unit, or the number of times the unit is contained in a quantity.

The denominator 1 is seldom written. (43).

131. The numerator of a fraction may be regarded as a whole number, the size of whose unit is indicated by the denominator.

Thus, $\frac{3}{4}$ of a dollar are three units of 25 cents value each, and five pence are $\frac{5}{12}$ of a shilling.

132. To add vulgar fractions; reduce them to a common denominator, (125) and (129), and add the numerators, writing the sum over the common denominator.

For the numerators are whole numbers or integers of the same unit value. (131).

133. To subtract vulgar fractions; reduce them to a common denominator, and write the difference of the numerators over the common denominator.

See (132).

134. To multiply a vulgar fraction; multiply the numerator. (131). Or, divide the denominator.

This last direction depends upon a principle hinted at in (117), the truth of which may be shown from (36) or from (65). For the identity of the cases in (36) and (65), compare (122), (75). If, after introducing a factor into one term of a fraction, we suppress it in both terms, the result is the same as though we had simply suppressed it in the other. For instance, take $\frac{3}{4}$, and by the factor 2 we may change it into the equal forms $\frac{6}{4}$ or $\frac{3}{2}$, and by the factor 3 into $\frac{1}{4}$ or $\frac{3}{12}$.

135. To multiply by a vulgar fraction; multiply by the numerator, and divide by the denominator. (120, or 121, 100, 37).

See also (103). In fact, by (100), this operation is one in what has been commonly called the "rule of three."

136. **To multiply vulgar fractions together: Multiply the numerators together for a new numerator, and the denominators for a new denominator.** (134, 135).

The numerator of the multiplicand is the number, (131), to be multiplied by the fraction which is multiplier, that is, (135), multiplied by its numerator and divided by its denominator, both of which conditions of (135) are fulfilled in (136), by (118, 119) or (120).

137. **To divide a vulgar fraction: Reverse the operation of (134).** See (55).

To divide by a vulgar fraction: Invert its terms and multiply.

For, by (73) and (75), we need but change the signs of our exponents, and multiply; that is, make our divisor (denominator) a dividend (numerator), and our dividend a divisor.

Or, again, by (107), we have, the ratio of the fraction to unity is that of the dividend to the quotient. And, by (121, 37), the product of the denominator and dividend equals that of the quotient and numerator. Or, the quotient equals the dividend multiplied by the denominator and divided by the numerator. See remarks under (135).

138. Examples for (129, 132 – 137).

1. Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{9}$, $\frac{5}{6}$, $\frac{7}{18}$, 3, and $\frac{4}{7}$ to their least common denominator.

$\frac{63}{126}$, $\frac{84}{126}$, $\frac{70}{126}$, $\frac{105}{126}$, $\frac{49}{126}$, $\frac{378}{126}$, $\frac{72}{126}$.

2. What is the sum of $\frac{1}{2}$, $\frac{5}{9}$, and $\frac{4}{7}$?

$\frac{63}{126} + \frac{70}{126} + \frac{72}{126} = \frac{205}{126} = 1\frac{79}{126}$.

3. When Ned had spent $\frac{7}{9}$ of his patrimony, what portion of his father's estate did he own, if his father left him $\frac{17}{33}$ of it?

$$\tfrac{17}{33} \times \tfrac{2}{9} = \tfrac{34}{297}.$$

4. Ned received $\frac{17}{33}$ of his father's estate, Dick $\frac{19}{45}$, and Bet the difference between them. What had the father left for other bequests?

$1 - (\frac{255}{495} + \frac{209}{495} + \frac{46}{495}) = -\frac{15}{495} = \frac{-1}{33}$. He had bequeathed more than his whole estate by a thirty-third part.

5. 7oz. of quicksilver cost 6s., how many £. is that a lb.?

$$\tfrac{6}{20} \div \tfrac{7}{16} = \tfrac{6}{20} \times \tfrac{16}{7} = \tfrac{6}{5} \times \tfrac{4}{7} = £\tfrac{24}{35} \text{ per lb.}$$

6. Perform the following operations.

$(\frac{3}{5} + \frac{2}{9}) \div \frac{1}{7} = \frac{259}{45}$; $\frac{3}{5} + (\frac{2}{9} \div \frac{1}{7}) = \frac{97}{45}$; $\frac{111}{108} - \frac{111}{112} = \frac{111}{3024}$.

139. In the following chapter we resume the subject hinted at in (4), and partly unfolded in (117).

CHAPTER VII.

DECIMAL FRACTIONS.

140. As the denominator 1 is not generally written, neither is 1 with zeros annexed. The numerators of such fractions are written by a law developed from (3), each figure indicating parts of

the units tenfold smaller at each remove to the right.

141. A point, called the decimal point, separates the integers from such fractions, which are called decimal fractions.

Thus, $2\frac{1}{10}$ is written 2·1, and $2\frac{1}{100}$ is written 2·01, &c.

142. The rules of Book I. evidently apply to all numbers decimally written, whether fractional or integral.

143. In addition and subtraction of decimal fractions, we must be careful to place tenths over tenths, &c., in a manner analogous to (7, 13, 93, 95, *b*).

Even in (132, 133) the same analogy can be seen.

144. In multiplication, the product must contain as many (fractional) decimal places as both the factors.

For instance, $2{\cdot}5 \times 2{\cdot}11 = 5{\cdot}275$.

Which follows from the law of (140), since the smallest fraction in the product must be the product of the smallest in the factors, and the tenth of a tenth is a hundredth, &c.

145. Conversely, by (55), the decimal places of the dividend must equal in number those of the quotient and divisor.

If the places in the divisor exceed those of the dividend, it must be owing, by (144), to zeros in the dividend, at the right of the significant figures, which have been dropped. Hence,

In applying this as a rule for fixing the decimal point of a quotient, places on the left of units in the quotient may be called negative decimal places; but zeros must be annexed to the dividend before division, enough to make its significant figures divisible by those of the divisor.

For a decimal denominator is a factor with a negative exponent. But, as by (14) the negative of a negative is positive, so a decimal factor [10, 100, 1000, &c.] may be called a decimal denominator with a negative exponent.

146. Examples for (141 – 145).

1. $2{\cdot}5 + {\cdot}03 + 3{\cdot}106 + 5{\cdot}001 = 10{\cdot}637.$
2. ${\cdot}001 + {\cdot}000001 + 6{\cdot}666 = 6{\cdot}667001.$
3. ${\cdot}001 - {\cdot}000001 = {\cdot}000999.$
4. $1{\cdot}01 - {\cdot}00007 = 1{\cdot}00993.$
5. $1{\cdot}01 \times {\cdot}00007 = {\cdot}0000707.$
6. ${\cdot}001 \times {\cdot}000001 = {\cdot}000000001.$
7. ${\cdot}001 \div {\cdot}000001 = 1000.$
8. ${\cdot}000001 \div {\cdot}001 = {\cdot}001.$
9. ${\cdot}003 \div {\cdot}00025 = 12.$
10. ${\cdot}003 \div {\cdot}00008 = 37{\cdot}5.$
11. ${\cdot}003 \div {\cdot}00009 = 33{\cdot}33333$ &c.
12. ${\cdot}07 \div {\cdot}00022 = 318{\cdot}1818$ &c.

147. **To reduce a vulgar fraction to a decimal;**

annex zeros to the numerator and divide by the denominator.

For this gives us a number which bears the same ratio to 10, 100, 1000, &c., (according to the number of zeros annexed,) that the numerator does to the denominator. That is, it gives us the numerator of a decimal fraction. See (121, 100, 135), and (140).

By this rule, $\frac{3}{4}$ becomes ·75, by annexing two zeros, and $75 : 100 = 3 : 4$. Again, $\frac{1}{125} =$ ·008 by annexing three zeros, and $8 : 1000 = 1 : 125$.

148. Whenever a divisor which is prime to its dividend contains other primes than 2 and 5, the quotient cannot be fully expressed in ordinary decimals.

For, by (49), the annexing of zeros cannot render it divisible by any factors except 2 and 5, since $10 = 2 \times 5$, $100 = 10^2$, &c.

149. The quotient in (148) will consist of a limited number of figures recurring perpetually in the same order, sometimes preceded by fixed figures.

For the divisor is a limited number, and we must, therefore, in dividing with zeros annexed, come at length to a remainder equal to some previous remainder, and the quotient figures will then begin to repeat themselves.

Thus, divide 3 by 77, and the quotient will be ·038961, and then a remainder of three will be left, so that these six places will recur constantly, and $\frac{3}{77}$ = ·038961038961038 &c.

150. Such quotients are called circulating decimals; or, if preceded by fixed figures, mixed circulating decimals.

The figures which are repeated are called the circulate. A circulate of one place is called a repetend.

151. A circulate is designated by a dot over its first and last places, and thus the quotients of all numbers may be written in decimal form.

And, as incommensurable quantities cannot be expressed by vulgar fractions, (123), we may, by (151), express in decimal form all quantities capable of expression by numbers.

152. Circulating and mixed decimals are similar and conterminous when the circulate begins and ends in each of them at the same distance from the decimal point. (141).

Thus, the mixed decimals ·270303 &c., ·171212 &c., ·000101 &c., which by (151) are written $\cdot 27\dot{0}\dot{3}$, $\cdot 17\dot{1}\dot{2}$, $\cdot 00\dot{0}\dot{1}$, are similar and conterminous; as are also the circulating decimals $\cdot\dot{0}3896\dot{1}$, $2\cdot\dot{1}7813\dot{1}$, $43\cdot\dot{0}0010\dot{2}$.

153. To make any numbers or fractions similar and conterminous.

Zeros annexed may be represented as circulates of any number of places, and thus whole numbers are brought into the form required, and likewise are decimals of fixed places.

And, since a circulating decimal may be conceived as beginning to circulate at any point, and the circulate to consist of any multiple of its real number of places,

Find the least common multiple of the numbers of places in the circulates. Extend the circulate which begins in the lowest place to a number of places equal to this multiple, and make the others conform to it.

For example, make 7, $1{\cdot}\dot{6}$, $1{\cdot}\dot{0}\dot{1}$, $2{\cdot}374\dot{2}\dot{1}$, and $3{\cdot}7\dot{8}2\dot{9}$ similar and conterminous. The places of the different circulates are in number, 0, 1, 2, 2, and 3; their least common multiple is 6. The circulate $\dot{2}\dot{1}$ begins lowest in the decimal scale. And the numbers in their new form will be $7{\cdot}000\dot{0}0000\dot{0}$, $1{\cdot}666\dot{6}6666\dot{6}$, $1{\cdot}010\dot{1}0101\dot{0}$, $2{\cdot}374\dot{2}1212\dot{1}$, and $3{\cdot}782\dot{9}8298\dot{2}$.

154. To discover what form of decimals a vulgar fraction will assume. (147).

a. Reduce it to its lowest terms. (128). Whichever factor 2 or 5 occurs in the denominator with the higher exponent, it indicates by its exponent the number of fixed places in the decimal.

For every couple of these factors 2 and 5, that is every 10, in the denominator must produce one, and only one, decimal place, and every other factor, 5 or 2, must also produce one, and only one, decimal place.

b. If other factors occur, the decimal will have a circulate, (148, 149, 150), and the number of its places may be discovered by dividing 1 with zeros annexed by these factors, until a remainder of 1 appears.

For the circulate must be formed by multiples of a circulate so made, and consist of the same number of places.

For instance, what will be the decimal form of $\frac{286}{9680}$?

$\frac{286}{9680} = \frac{13}{440} = \frac{13}{2^3.5.11}$, whence there are three fixed places in the decimal; and $\frac{1}{11} = \cdot\dot{0}\dot{9}$, whence there is a circulate of two places.

Let us try it; $\frac{13}{2} = 6{\cdot}5$; here is one of the fixed places. $\frac{6{\cdot}5}{2} = 3{\cdot}25$; here is the second. $\frac{3{\cdot}25}{2} =$ 1·625; here is the third. Now division by 5 cannot increase the number of places, since the decimal ends in 5. (64, *b*). If we had first divided by 5, the division by the first 2 would not have increased the decimal places. (64, *a*).

$\frac{1{\cdot}625}{5} = {\cdot}325$, and, finally, $\frac{{\cdot}325}{11} = {\cdot}029$, with a remainder of 6, which will give a circulate of $6 \times \cdot\dot{0}\dot{9} = \dot{5}\dot{4}$. So, $\frac{13}{440} = \cdot 029\dot{5}\dot{4}$.

155. The circulating part of a decimal is produced from a fraction equal to the circulate divided by as many places of nines as the circulate contains, with as many zeros annexed as there are fixed decimal places.

For, $\frac{1}{9} = \cdot\dot{1}$, and $\frac{1}{99} = \cdot\dot{0}\dot{1}$, $\frac{1}{999} = \cdot\dot{0}0\dot{1}$, &c., and other circulates are simply multiples of these. And the removal of the circulate to the right is equivalent to annexing zeros to the denominator, by (140, 137).

For instance, $\cdot 29\dot{5}\dot{4} = \cdot 29 + \frac{54}{9900}$, and $5\cdot\dot{1}\dot{7} = 5\frac{17}{99}$.

156. To reduce a finite decimal to a vulgar fraction. Write under it 1 with zeros annexed, as many as the places of the decimals; reduce it by (128).

The greatest common measure must consist wholly of 2's and 5's. Hence, by (64, *a*, *b*), the reduction can be performed rapidly, without seeking the greatest common measure.

157. To reduce a circulating or mixed decimal to a vulgar fraction, the articles (155, 156, 132, 128) are sufficient.

158. To add circulating decimals: Make them similar and conterminous. Divide the sum of the circulates by so many places of nines; the remainder will be the circulate of the sum, (zeros prefixed if necessary,) and the quotient is to be added to the fixed places.

This rule flows immediately from (155, 132).

Examples. $1\cdot\dot{6} + 3\cdot\dot{7}\dot{0} + 5\cdot 0\dot{3}\dot{3} = 1\cdot 6\dot{6}\dot{6} + 3\cdot 7\dot{0}\dot{7} + 5\cdot 0\dot{3}\dot{3}$. The sum of the circulates is 106, whence $1\cdot 6\dot{6}\dot{6} + 3\cdot 7\dot{0}\dot{7} + 5\cdot 0\dot{3}\dot{3} = 10\cdot 4\dot{0}\dot{7}$.

159. To subtract circulating decimals: Make them similar and conterminous, and, if the circu-

late of the subtrahend is larger than that of the minuend, subtract 1 from that of the minuend, and proceed as in (13).

If the minuend circulate is smaller than that of the subtrahend, you borrow by (13) one from the next place above it, and count it as tenfold greater than if it stood in the highest place of the circulate. But it is not so great in reality, for the circulate places have a higher value than decimals, their denominator being a row of 9's, and not 1 with zeros annexed. (155). You thus add to the circulate 10 instead of 9, 100 instead of 99, &c., and must subtract 1 in order to preserve its true value.

$$11\cdot\dot{0}12\dot{3} - 9\cdot\dot{0}2\dot{3} = ? \qquad \begin{array}{r} 11\cdot\dot{0}1230123012\dot{3} \\ 9\cdot\dot{0}2302302302\dot{3} \\ \hline 1\cdot989278207099 \end{array}$$

160. To multiply circulating decimals: Change them to the form of vulgar fractions, (157), multiply, (136), and reduce the product to decimal form. (147).

We here lose the decimal *form* boasted of in (151), but the reality of decimal notation is lost so soon as the circulate is marked with dots, for its denominator is then no longer a power of 10, but a multiple of 9.

$$7\cdot1\dot{3} \times 5\cdot\dot{4} = \tfrac{642}{90} \times \tfrac{49}{9} = \tfrac{31458}{810} = \tfrac{3145\cdot8}{81} = \tfrac{349\cdot5\dot{3}}{9} = 38\cdot83\dot{6}.$$

When only one factor circulates, (158) may be used, since multiplication is addition.

161. To divide circulating decimals: Change

them to the form of vulgar fractions, divide, (137), and reduce the quotient to decimal form.

$8{\cdot}\dot{1}\dot{9} \div 15{\cdot}\dot{0}\dot{1} = \frac{811}{99} \div \frac{1486}{99} = \frac{811}{99} \times \frac{99}{1486} = \frac{811}{1486} = {\cdot}5\dot{4}57604306$ &c.

This example shows the boast of (151) to be practically useless, since the circulate in the quotient, beginning at the second place, may extend to the 744th place, which renders it practically no circulate at all.

162. Circulating decimals are practically to be distinguished from others, only when the circulate consists of few places.

For, in most practical questions, figures are not required more than 6 or 8 places from the decimal point on either the right hand or the left. Quantities requiring higher decimal notation than this are considered as, on the one hand, boundless, or, on the other, as small enough to be neglected, and the ratio of such quantities belongs not to arithmetic, but to infinitesimal calculus.

CHAPTER VIII.

POWERS AND ROOTS.

163. Since the square, whose side is a linear unit, is taken as a unit of superficies, (88, *e*), the second power gives the area of a square whose side is represented by the first. And, since this is the most common interpretation, (85), of the second power, that power is in mixed arithmetic called the square, and the second root is called the square root.

Another interpretation of powers will be given in (174).

164. To find the square root of a given number.

The rule may be derived directly from (78).

Divide the given numbers into periods of two figures each, beginning at the decimal point. (78, *a*). From the first period subtract the largest square it contains, setting down its root as the first figure of the required root. (78, *b*). To the remainder annex the next period. Divide the remainder by double the root already found, (78, *c*), the quotient will give the next figure of the root. Annex the new root figure to double the previous figure and multiply by the new figure. Subtract this product from the previous remainder, (78, C and *c*), and treat the remainder as before.

Or, the same rule may be derived from the inspection of a square representing the number. Let the number, for instance, be 15625. Now, it is at first sight plain, that the square root is between 100 and 200. Imagine, then, a square arranged of 15625 smaller squares or units. From it subtract a square whose side is 100. This leaves an L whose contents are 5625. Now the legs of this L are the product each of their length, 100, by the width of the legs; whence we may obtain a number expressing nearly that width, by dividing half of 5625 by 100, which gives 28·125. But let us, for the sake of ease in working, call it in round numbers 20, and subtract the legs, 2000 each, from 5625. But, to take out a complete L, we must also subtract the corner square of $20 \times 20 = 400$, which leaves another L containing 1225, the inner side of whose legs is in length 120, and the quotient $\frac{1225}{2 \times 120}$ must express nearly their width. That quotient is 5·1 &c. Call it in whole numbers 5, and, subtracting the legs, 5×120 each, and the corner square, $5 \times 5 = 25$, we find zero remaining; whence the side of the square was 125; and our operation in obtaining it has been essentially that of the rule.

Or, the rule may be obtained from an examination of (78, C), and similar examples.

165. What is the square root of 12345·6789 ?

```
                  1 23 45 · 67 89 (111·11111060 &c.
                  1
               21) 23
                   21
              221) 245
                   221
             2221) 2467
                   2221
            22221) 24689
                   22221
           222221) 246800
                   222221
          2222221) 2457900
                   2222221
         22222221) 23567900
                   22222221
       2222222206) 13456790000
                   13333333236
```

What is the name as a number of $20164^{\frac{1}{2}}$?

```
         2 01 64 (142
         1
     24) 101
          96
    282) 564
         564
```

166. By adding periods of zeros the root may be extended to any degree of exactness.

But, since the square of no single figure ends in zero, the square root, which is incomplete when the significant periods of its power are exhausted, cannot be expressed in decimals. Neither will it result in a circulating decimal. For the square of a circulate must be a circulate, since the squaring of its denominator removes no prime factors from it. (148, 149).

167. The square root of a vulgar fraction is the square root of its numerator divided by the square root of its denominator. (136, 26, 76, 85).

Thus, $(\frac{4}{9})^{\frac{1}{2}} = 4^{\frac{1}{2}} \div 9^{\frac{1}{2}} = \frac{2}{3}$.

168. Square roots inexpressible in decimals are inexpressible in vulgar fractions.

For common factors in the terms of the fraction, (156, 157), cannot alter the value of the root, since both terms of the root would contain the root of the common factors. (65, 151).

169. The inversion of the rule in (164, 165) gives an easy method of squaring a number.

To square 142, for example.

By (22).
$$\begin{array}{r} 142 \\ 142 \\ \hline 284 \\ 568 \\ 142 \\ \hline 20164 \end{array}$$

By (169).
$$\begin{array}{r} 142 \\ 142 \\ \hline 1 \\ 96 \\ 564 \\ \hline 20164 \end{array}$$

To square 1829.

By (22).
$$\begin{array}{r} 1829 \\ 1829 \\ \hline 16461 \\ 3658 \\ 14632 \\ 1829 \\ \hline 3345241 \end{array}$$

By (169).
$$\begin{array}{r} 1829 \\ \hline 1 \\ 224 \\ 724 \\ 32841 \\ \hline 3345241 \end{array}$$

That is, each figure is multiplied by itself and by the double of the figures on its left; and each product occupies a position two places removed from the neighbouring products.

170. Since a cube whose edge is a linear unit is taken as the unit of solidity, (88, *f*), the third power expresses a cube whose edge is the first power. Hence the third power is called the cube; and the third root the cube root.

171. To extract the cube root of a given number.

a. Let us again cube 57, keeping the products of each figure distinct, but in their decimal places.

By (22).

$$\begin{array}{r} 57\\ 57\\ \hline 399\\ 285\\ \hline 3249\\ 57\\ \hline 22743\\ 16245\\ \hline 185193 \end{array}$$

By (171, *a*).

$$\begin{array}{cccc} & & 5 & 7\\ & & 5 & 7\\ \hline & & 5\,.\,7 & 7^2\\ & 5^2 & 5\,.\,7 & \\ \hline & 5^2 & 2\,.\,5\,.\,7 & 7^2\\ & & 5 & 7\\ \hline & 5^2\,.\,7 & 2\,.\,5\,.\,7^2 & 7^3\\ 5^3 & 2\,.\,5^2\,.\,7 & 5\,.\,7^2 & \\ \hline 5^3 & 3\,.\,5^2\,.\,7 & 3\,.\,5\,.\,7^2 & 7^3 \end{array}$$

$$\begin{array}{rcr} 5^3 & = & 125\\ 3\,.\,5^2\,.\,7 & = & 525\\ 3\,.\,5\,.\,7^2 & = & 735\\ 7^3 & = & 343\\ \hline & & 185193 \end{array}$$

It is evident from this example, that

After obtaining two figures of the root by (78), we may, instead of cubing both figures, subtract from the first remainder and the annexed second period thrice the first figure squared multiplied by the second, thrice the second squared multiplied by the first, and the cube of the second, giving each product its proper decimal place, and the remainder will be a second remainder with which to proceed as with the first.

The same rule may be obtained, as in (164), by representing a cube containing 185193 unit cubes. Subtract at first a cube whose edge is 50; and we have a three sided cap left, containing 60193. Divide the third part of it by 50^2, and the quotient approaches an expression of its thickness. Calling the thickness 7, a little less than the quotient, we subtract $3 \times 2500 \times 7$, for the three flat slabs on the sides of the cube; $3 \times 50 \times 7^2$, for the narrow slabs at the edges, and 7^3, for the corner. Thus we have performed the operation of the rule.

What number expresses $187149248^{\frac{1}{3}}$?

$$\begin{array}{rr|l}
 & & 187\ 149\ 248\ (572 \\
5^3 = & & 125 \\
\cline{3-3}
3 . 5^2 = & 75 & 62149 \\
\cline{1-2}
75 \times 7 = & 525 & \\
3 . 5 . 7^2 = & 735 & \\
7^3 = & 343 & 60193 \\
\hline
3 . 57^2 = & 9747 & 1956248 \\
\cline{1-2}
9747 . 2 = & 19494 & \\
3 . 57 . 2^2 = & 684 & \\
2^3 = & 8 & 1956248 \\
\cline{3-3}
\end{array}$$

In dividing 621 by 75, the quotient is 8; but this would make too large a subtrahend, after 15 times its square was added.

172. In finding the cubical contents of bodies, the notation of (88, *d*) is sometimes retained in both factors. The foot is considered as a unit, and the smaller denominations as fractions; and the multiplication of the denominators is, as in decimals, indicated by the position of the numerators. This is sometimes called duodecimals, though it is a mixture of the decimal and duodecimal scales. For true duodecimals see (188).

For example: A load of wood 7ft. 7in. long, 3ft. 9in. wide, and 4ft. 3in. high, contains how many feet?

7	7			
3	9			
22	9			Product by 3.
5	8	3		Product by $\frac{9}{12}$.
28	5	3		
	4	3		
113	9	0		Product by 4.
7	1	3	9	Product by $\frac{3}{12}$.
120	10	3	9	

That is, $120\frac{10}{12} + \frac{3}{1728} + \frac{9}{20736}$ of cubic feet, or, $(12 \times 120 = 1440$, and $12 \times 1450 = 17400,)$ it equals $17403\frac{9}{12}$ cubic inches.

Multiplication, in other mixed notation, is usually capable of interpretation only by (101); and the conditions of the problem must show what is to be taken as the unit of the multiplier. Thus, as three pence times four pence cannot mean $\frac{1}{12}$ of a square shilling, it depends upon the conditions of the problem whether it signifies 3 times four pence, or $\frac{3}{12}$ of four pence, or $\frac{3}{240}$ of four pence, according as penny, shilling, or £. is the unit.

173. Articles (166 – 168) will apply also, with the necessary changes, to the cube root.

174. Other powers and roots than the second and third do not seem to admit easily of geometrical interpretation; and the general interpretation of the power of a quantity must be, "the same quantity, increased in a ratio discoverable by means of the exponent, though not indicated by it."

The interpretations of (163, 170) make the second and third roots an entirely different species of quantity from their powers; and (172) interprets a product as a distinct species of quantity from the multiplicand; — multiplication changing linear into superficial, and superficial into cubic measure. But this is contrary to the usual signification of multiplication, which means an enlargement of the quantity without change in its character. (102). And powers are simply multiplications, the factors being indicated in a peculiar manner. (26, 27, 76, 77).

And, as factors are properly numbers, (101), and yet, in mixed arithmetic, are quantities, or the ratios of those quantities to unity, (100), so, since exponents take fractional forms, (76), quantities, or their ratio to their unit, may be used as exponents. And, as these exponential ratios may be of every value, it follows, that

175. Any quantity may be represented either as the product or as the power of any other quantity of the same species.

Let us take, for instance, 2 and 5. Now it is manifest, that $2 = 5 \times \frac{2}{5}$ and $5 = 2 \times \frac{5}{2}$. So they are products of each other by their mutual ratio.

Let us see what power of 2 is equal to 5. Since $2^2 = 4$ and $2^3 = 8$, it appears, that the exponent must be more than 2 and less than 3. Let us try $\frac{7}{3}$. Raise 2 to the 7th power and it equals 128, and $128^{\frac{1}{3}} = 5{\cdot}03$. Whence it appears, that 5 is a power of 2 whose exponent is nearly $\frac{7}{3}$.

The whole system of logarithms (the invaluable gift of Napier to the world) is based on (175).

176. To extract a fractional root or raise to a fractional power: Raise to the power expressed by the numerator and extract the root indicated by the denominator, or reverse the process.

For instance, what number expresses $27^{\frac{2}{3}}$? Here $27 = 3^3$, whose square is 3^6, whose cube root is $3^2 = 9$. Or, the cube root of 27 is 3, whose square is 9.

But, in this example, as in all examples of fractional exponents to which answers in exact numbers are possible, we gain the result more easily by resolving the number into prime factors, (74), and acting upon (29, 135). For instance, what is the value of $7056^{\frac{3}{2}}$? $7056 = 2^4\,3^2\,7^2$, whence $7056^{\frac{3}{2}} = 2^6\,3^3\,7^3 = 592704$.

What is the 100th power of that number whose 101st power is 37? This I find by other than arithmetical rules to be 35·70058 nearly. But the labor of calculating $37^{\frac{100}{101}}$ by arithmetic wonld be too mighty to attempt. Indeed, fractional exponents whose terms are larger than 3 are apt to be beyond the practical range of arithmetic.

When the numerator is more than half the denominator, the process is made easier by finding the root corresponding to the exponent, which, added to the given exponent, would make 1, and dividing the number by this root. For the law of (72) applies to fractional exponents.

CHAPTER IX.

APPROXIMATION.

177. Division and extraction of roots are but processes of successive trials and approximation; and they introduce themselves in the vast majority of arithmetical problems.

For instance, would you divide 257 by 43, you first guess at a quotient of 6, but on trial find it too large, and then try 5, and find then there is a remainder, and that the division must introduce fractions. So you add 9 tenths to the quotient, but still there is a remainder; then you add 8 hundredths and find it is too much; and then you change it to 7 hundredths, and so go on, but never attain an exact result.

178. Since we cannot avoid methods of approximation in arithmetic, we may extend their use to other problems, without feeling that we are lowering the standard of the science.

From some reason, apparently from a notion that arithmetic should use no rules which do not give exact results, the following most useful rule, called in the old books the Rule of Double Position, has been omitted in most of the modern school-books.

179. The most general method of arriving at approximate results in arithmetic is, to take for granted that the errors of results are proportioned to the errors of supposition. If they are, (37) gives us the means of discovering our errors of supposition; if they are not, it at least gives us the means of estimating that error, and successive trials will at length discover it.

That is, make two suppositions, and suppose that the difference of their results is to the error of one of them as the difference of the suppositions is to the error of one of them.

For instance, to find the cube root of 185193. Suppose 60 and suppose 50. Their difference is 10, the difference of their cubes is 91000, and the error of the cube of 50 is 60193. So, by (179) and (37), the error of 50 is $\frac{601930}{91000}$, which is almost 7. On trying 57, we find its cube is the given number, 185193.

What number is it whose cube added to its square equals 140? Suppose 4 and suppose 5, and add their squares to their cubes, the results will be 80 and 150. Hence the error of 5 is to 1, as 10 is to 70, and the true number appears to be $4\frac{6}{7} = \frac{34}{7}$. But $(\frac{34}{7})^2 + (\frac{34}{7})^3 = 138.1$, and $4\frac{6}{7}$ is too small by about $\frac{1}{6}$ of its difference from 5, and the true number seems to be $4\frac{37}{42}$. Thus each trial will give a number nearer and nearer to the exact truth.

CHAPTER X.

PRACTICAL HINTS.

180. As every process in arithmetic must be performed mentally before it is written, the writing of the operation usually retards the rapidity of its execution.

Save that it sometimes aids in rapidity of execution by its assistance to the memory. Indeed, the assistance to the memory is the only object of using the pencil. For if a man can solve a problem in his head, he uses no paper.

The student is earnestly recommended to devise ways of rendering any problem, which may be presented, capable of solution without the pencil, and to practise constantly the resolution of questions by mental processes only.

Of course, no particular rules can be given, since the means will differ in every example. The following cases will, however, show to what we allude; and it will be noticed, that in the form here given no use of the pencil is necessary; the changes and operations being such as can easily be performed mentally.

$8649 + 7987 = 8649 + 8000 - 13 = 16636.$

$7234 + 1991 + 1809 = 7034 + 2000 + 2000 = 11034.$

$1827 + 5648 + 4349 = 1827 + 10000 - 3 = 11824.$

$512347 + 287452 = 800000 - 201 = 799799.$

$16636 - 7987 = 16636 - 8000 + 13 = 8649.$

Other operations may shorten labor a great deal by rendering but few figures necessary. In the following examples, although we have written them with many figures, yet in most cases very few figures are necessary. Small denominators, zeros at the end of a number, &c., would not be actually written in practice, and thus the following examples would be performed with very few figures; frequently none being required.

$57 \times 13\frac{3}{4} = \frac{5700}{8} + 57 + \frac{57}{4} = 712\frac{1}{2} + 57 + 14\frac{1}{4} = 783\frac{3}{4}.$

$876 \times 375 = (\frac{876000}{8} \times 3) = 328500.$

$8178567891 \times 5 = \frac{81785678910}{2} = 40892839455$

$78964 \times 125 = \frac{78964000}{8} = 9870500.$

$856 \times 128 = 107000 + 2568 = 109568$

See, also, (106). The following examples in division may suggest some methods of saving labor. When you divide by factors of a divisor successively, divide first by those which contain only powers of 2 and 5.

$1703 \div 28 = 425\cdot75 \div 7 = 60\cdot82\dot{1}4285\dot{7}.$

$167784 \div 55 = 33556\cdot8 \div 11 = 3050\cdot6\dot{1}7\dot{2}.$

$17645 \div 125 = (17645 \div 10^3) \times 8 = 141\cdot16.$

Before performing any operations, be sure that there are no factors by which you will both multiply and divide; for such factors may be omitted.

For instance, how many yards of carpeting that is $\frac{3}{4}$yd. wide will cover a room 18ft. by 60? Since it takes 9ft. to make a square yard, (88, *e*), and the carpeting is $\frac{3}{4}$yd. wide, we simply multiply 4, 2, and 20 together, giving us 160 yards' length of carpet.

$$\tfrac{1}{2} \times \tfrac{2}{3} \times \tfrac{3}{4} \times \tfrac{4}{9} \times \tfrac{7}{16} \times \tfrac{8}{5} \times \tfrac{2}{7} = \tfrac{1}{9} \times \tfrac{1}{5} = \tfrac{1}{45}.$$
$$1737 \div 63 = 193 \div 7 = 27\tfrac{4}{7}.$$

Especially would we recommend him to attain a habit of quickly detecting common factors in the terms of ratios, by the omission of which the solution of many problems is rendered very easy.

Several treatises on arithmetic have been published in which this principle of suppressing common factors, (66), is made the main engine of operation.

To enable the reader more readily to detect prime factors, we shall offer a few remarks.

The factor 2 may always be detected by (64, *a*), and its square by the last two places being divisible by 4; its cube by the last three places being divisible by 8.

The factor 3 and its square are detected by (64, *d*).

The factor 5 is known by (64, *b*, *c*).

The factor 7 may generally be detected in one of the following ways. When the units of a sum are equal to half the tens, it must of course be a multiple of 7, since the remainder from every 20 must be 6 and a unit will make it 7. If the number of tens be odd, or if from other causes this test of the factor 7 does not appear, it may sometimes be made apparent by subtracting or adding 7, 7 tens, or any convenient multiple of 7. Thus, 105, 147, 126, 189, &c., are known as multiples of 7, because 5, 7, 6, 9 are halves of 10, 14, 12, and 18. And 133, 154, 182, &c., are recognized as multiples of 7 by the addition or subtraction of 7 bringing the same result.

Again, in dividing hundreds by 7 there is a remainder of 2 in each hundred, and if the number in the two right-hand places gives us a 5 for each hundred, or 1 for every three hundred, the number is a multiple of 7. Thus, 1155 is a multiple of 7, since $\frac{55}{11} = 5$, and 6923, because $\frac{69}{23} = 3$.

And, since 1001 is a multiple of 7, all multiples of it are ; such as 7007, 38038, 789789, &c.

The factor 11 may always be known when the sum of the figures in the odd places is equal to, or differs by 11 from the sum of the figures in the even places. For multiplication by 11 causes both the odd places and the even places to contain every figure of the multiplicand, and the addition of the product can only disturb this by taking 10 from one place and adding it as 1 in the next place, making a difference of 11. For instance, 9493 is a multiple of 11, because $9 + 9 = 4 + 3 + 11$; and 747439, because $7 + 7 + 3 = 4 + 4 + 9$, &c.

The factor 13 is found in multiples of 1001, and in those numbers in which there are four odd ones to each hundred, such as 624, 728, 1352, 1768, &c.; in which $4 \times 17 = 68$, $4 \times 6 = 24$, &c.

Other ways of detecting the presence of a factor may be discovered by the student. All these are at times of service, while at other times the readiest way of detecting a factor is to attempt division by it.

In division, the work may be written with fewer figures than usual, and in much smaller space, if we write the quotient over the dividend and do not carry down the dividend figures to a line with the remainders. In this case, place the first quotient figure over its own product, and the decimal point in the quotient will fall as many places to the right of that in the dividend as there are decimal (fractional) places in the divisor.

For instance, to divide 1·08062 by 31, the work may be written thus:

$$\begin{array}{r} 31 \\ \cdot03485870967 \\ \hline 1\cdot08062000000 \\ 93485877967 \\ \hline 15687232143 \\ 1245412181 \\ \hline 212200022 \\ 21220002 \\ \hline \end{array}$$

While, by the ordinary method of writing, it would require a space more than half as wide again, and nearly 2½ times as long. And one line of figures here written is unnecessary, as the figures 21220002 might merely, in practice, have a dot put under them.

The first quotient figure, 3, is placed over the last figure of its product, 93. The decimal points are made to coincide, because there are no decimal fractions in the divisor. Then, 108 — 93 = 15, and 4 × 31 = 124. Of this product the 4 is placed under the dividend figure 0, and the 12 under the remainder 15. Then, 150 — 124 = 26; of which remainder 6 is on a line with 15, and 2 is beneath the 12. Again, 8 × 31 = 248; of which product, 8 is written under the dividend figure 6; 4 is put under the last remainder 6, and 2 under the remainder 2; (instead of which, in practice, we would simply put a dot under the remainder 2;) and so on with the rest.

This plan for division affords so easy a method of decimal pointing, and is so neat and compact in form, as to deserve a more general use.

Solve examples 7, 9, 10, and 12 of (146) by this method.

$$\begin{array}{r} \cdot000001 \\ 1000\cdot \\ \hline \cdot001 \\ \cdot \end{array} \qquad \begin{array}{r} \cdot00025 \\ 12\cdot \\ \hline \cdot00300 \\ 250 \\ \hline 5 \\ \cdot \end{array} \qquad \begin{array}{r} \cdot00008 \\ 37\cdot5 \\ \hline \cdot003000 \\ 2460 \\ \hline 64 \\ \cdot\;\cdot \end{array} \qquad \begin{array}{r} \cdot00022 \\ 318\cdot\dot{1}\dot{8} \\ \hline \cdot070000 \\ 66262 \\ \hline 484 \\ 272 \\ \hline 1 \\ \cdot \end{array}$$

Divide 1375612·358716 by 367·12.

```
367·12
          3747·0373
1375612·358716
1101364843646
 27425851517
 25698481983
  172637052
  146890601
   258373
   ··6151
    121
    · · ·
```

In proving the correctness of our work in multiplication and division, we may remember that in (171, *a*) the only product not a multiple of 5 was 7^3. In like manner, if any multiple of 5 plus any excess had been multiplied by any multiple of 5 plus any excess, the only product not a multiple of 5 would have been the product of one excess by the other. And so of any number. Let then the factors be each divided by some number, and the remainders multiplied together. If this product and the product of the factors be now divided by that same number, they will leave equal remainders.

The numbers 9 and 11 recommend themselves for such divisors, by the ease with which remainders from their divisions can be discovered, and from the fact that every figure is used in discovering them, so as almost to exclude error.

For, the sum of the figures in a number, when divided by 9, leaves the same remainder as the number so divided; and if, from the sum of the odd places, increased by 11 if necessary, we subtract the sum of the even places, we obtain the remainder which would result from division by 11.

For instance, $849 \times 8367 = 7103583$. To prove which we observe, that $9 + 8 - 4 - 11 = 2$; $7 + 3 + 11 - 6 - 8 = 7$; $2 \times 7 - 11 = 3$; and $3 + 5 + 7 - 8 - 3 - 1 = 3$. Or, by means of 9's, $8 + 4 - 9 = 3$; $6 \times 3 - 2 \times 9 = 0$, and the sum of the figures in the product is also a multiple of 9.

The process of casting out nines is commonly taught, and when combined with the casting out of elevens, it forms an almost infallible test, very easily applied.

CHAPTER XI.

SYSTEMS OF NUMERATION.

181. The decimal system was probably suggested by the fingers of the hand.

In proof of which, all savage, as well as civilized men, count upon their fingers up to ten, and then renew the count.

182. The decimal system is, then, the natural system, but it is a useful exercise to practise in others.

Such practice enables one to separate more perfectly his idea of numbers from that of their names, and gives him a greater range of arithmetical thought.

183. The number by which a figure is multi-

plied in any system by removing it one place in notation is called the radix of the system.

Thus, 10 is the radix of the decimal or denary system, 12 of the duodenary, &c.

184. It needs only ordinary care in the student, to enable him to work in any system, by rules analogous to those given for the decimal system.

He need only keep constantly in mind the value of his radix.

185. To change numbers from one system to another: Divide the given number repeatedly by the radix of the new scale, and the successive remainders will be the figures in the successive places of notation.

See (94), in which this operation is illustrated. [After the words, " then the odd rows," in (94), add " next the odd piles."]

The operation may be reversed: Divide by the highest power of the new radix, the remainder by the next lower power, and so on; the quotients will then be the figures in the new notation.

186. To change fractions of the form (140, 141) to a new scale: Multiply by the radix of the new scale, and again multiply the fractional part of the product by the radix, and so on; the

whole numbers of the products will be the successive figures of the new fraction.

This is, by (137), perfectly analogous to (185).

187. In a binary system there would be but two characters, one representing zero, and the other 1, 2, 4, 8, &c., according to its place.

Calling, then, 10 a couple; 100 a second couple, or couple of couples, &c., we shall in a binary system read numbers by units, couples, second couples, &c., instead of units, tens, hundreds, &c.

Divide eleven by three in a binary system.

Eleven is a third couple, couple and one; three, a couple and one.

$$
\begin{array}{rl}
11) & 1011\,(11{\cdot}\dot{1}\dot{0} \\
 & \underline{11} \\
 & 101 \\
 & \underline{11} \\
 & 100 \\
 & \underline{11} \\
 & 10
\end{array}
$$

The above operation may be read thus; a couple and one into a second couple and one, once and a couple over. Bring down one, and it goes again the same. Place a binary point, (141), in the quotient, and annex zero to the remainder. A couple and one into a second couple, once and one over. Annex zero, and, as a couple and one is not contained in a couple, put zero in the quotient; but, as the remainder has now become the same as the preceding, it appears that the binary fraction will consist of the circulate $\cdot\dot{1}\dot{0}$. The value of this circulating binary, (155, 184), is $\frac{10}{11}$, equal to two thirds, as it should be for a quotient of eleven by three.

Multiply eleven by thirteen in a binary system.

$$\begin{array}{r l}
1011 & =(8+2+1=11).\\
\underline{1101} & =(8+4+1=13).\\
1011 & \\
1011 & \\
\underline{1011} & \\
10001111 & =(128+8+4+2+1=143).
\end{array}$$

Answer; a seventh, third, second, and first couple and one.

Divide sixty-nine by twenty-three.

$$\begin{array}{r r l}
10111) & 1000101 & (11=(2+1=3).\\
 & \underline{10111} & \\
 & 10111 & \\
 & \underline{10111} &
\end{array}$$

Divide seventy-three and six tenths by eighteen and four tenths.

See (186) as well as (185).

$$\begin{array}{r r l}
10010{\cdot}01\dot{1}00\dot{1}) & 1001001{\cdot}\dot{1}00\dot{1} & (100{\cdot}=(4)\\
 & \underline{10010{\cdot}01\dot{1}00\dot{1}} & \\
 & 0 &
\end{array}$$

The circulates here so fall, that the quotient is perfect without the need of (161). For the places of zeros in the quotient, see (145), and examples 7, 9, 10, 11, and 12 of (146).

188. In a duodenary system, eleven characters will be necessary beside the zero.

Let us take the 9 figures, and for 10 and 11 use *t* and *e*. The denominations will be units, dozens, gross, great gross, &c.

Multiply seventy-one by one hundred and twenty-two, that is, five dozen and eleven by ten dozen and two.

$$\begin{array}{r} 5e \\ \underline{t2} \\ et \\ \underline{4e2} \\ 501t \end{array} \qquad \text{The same with factors reversed,} \Big\} \qquad \begin{array}{r} t2 \\ \underline{5e} \\ 93t \\ \underline{42t} \\ 501t \end{array}$$

The operation may be read thus. Twice eleven is a dozen and ten; set down *t* and carry one; twice five is ten and one makes *e*. Then ten times eleven is nine dozen and 2, set down 2 and carry nine; ten times five is four dozen and 2, and 9 makes 4 dozen and *e*. Add up, and the amount is five great gross, one dozen, and ten, $= 5 \times 1728 + 12 + 10 = 8662 = 71 \times 122$.

Divide $998e65$ by $et7$.

$$\begin{array}{l} et7)\,998e65\,(9te \\ \underline{8te3} \\ \,t986 \\ \,\underline{9t9t} \\ \,tt85 \\ \,\underline{tt85} \\ \,0 \end{array}$$

Multiply three elevenths by five dozen-and-four'ths.

$$\frac{3}{e} \times \frac{5}{14} = \frac{13}{128} = \cdot 10\dot{3}.$$

Change $\cdot 10\dot{3}$ duodecimal to decimal form. (186), (160), (158).

$$\begin{array}{r} .10\dot{3}\\ t\\ \hline 0{\cdot}t2\dot{8}\\ t\\ \hline 8{\cdot}63\dot{3}\\ t\\ \hline 5{\cdot}28\dot{8}\\ t\\ \hline 2{\cdot}33\dot{3}\\ t\\ \hline 2{\cdot}88\dot{8}\\ t\\ \hline 7{\cdot}33\dot{3} \end{array}$$

So the fraction in decimal form is ·0852$\dot{2}\dot{7}$. Here, by (158), the radix of the first product in each multiplication is eleven, and in the succeeding places twelve. That $\dot{2}\dot{7}$ will be a circulate in the decimals appears from the recurrence of the mixed, (150), duodecimal ·33$\dot{3}$.

Now look at the fractions three elevenths and five sixteenths. Since sixteen is the fourth power of 2, and an eleventh is equal to ·0$\dot{9}$, the decimal of their product will have four fixed places, and two in a circulate, which ·0852$\dot{2}\dot{7}$ has. (154).

189. The student may amuse himself with examples in other systems. We have selected the binary and duodenary, because of the importance which Leibnitz attached to one, and the fact of the extensive use of the other.

Leibnitz thought the binary arithmetic peculiarly adapted to develope the properties of numbers; and its simplicity, beauty, and the harmonious appearance of numbers written in it, made it in his opinion a beautiful image of the creation of all things from nothing by the Infinite One, and an illustration of the saying in Scripture, "One thing is needful." For all numbers being written by 1 and 0 shows, says his hexameter,

"Omnibus ex nihilo ducendis SUFFICIT UNUM."

CHAPTER XII.

RELATION OF ARITHMETIC TO OTHER SCIENCES.

190. Arithmetic is a branch of the mathematics.

Though usually the first branch taught to a child, it perhaps holds that place unjustly. Forms are earlier objects of thought than numbers, and require less effort of abstraction to be comprehended; and this indicates that geometry in its simplest elements should be the first study in the mathematics.

191. Mathematics is the science of quantity.

Not of measurement, but of that which is capable of measurement. It may perhaps be defined as the science of the finite considered in its relations as finite. But since all measurable bounds (*fines*) are, at least in the present state of the human mind, those imposed by space and time, the laws of space and time include all the mathematics. Even those things which appear most removed from these elements, such as force and currency, (88, *a*, *i*, *j*, *k*, 89), are really measured by them. Mathematics may also be considered as an art and a language; (1); an art in its solution of problems, a language in its statement of them.

192. Arithmetic is the science of discontinuous quantity.

Numbers themselves are manifestly discontinuous, there being a break or discontinuity between each successive unit. And in the expression of quantities by the ratios of numbers, it is manifest there can only be the expression of discontinous quantities. For when a quantity, bearing to its unit the ratio of one number to another, begins to be increased, it must become incommensurable with its unit before it can bear to it the ratio of any other two numbers. Else numbers themselves are not discontinuous.

193. Mathematics are necessarily connected with all physical sciences.

For, since all the physical world is necessarily governed by the laws of space and time, the study of that world and of those laws must mutually aid each other. (191). And the truth of this remark is written on every page of the history of science. Astronomy has been the parent of mathematical sciences, and has been restored to youth and made immortal by her own children. So, too, with physical sciences of later growth. To mention no others, the science of light has proposed the most wonderful problems to the mathematician, and received answers which have placed optics by the side of astronomy, as the most glorious triumphs of the human intellect, and have opened to the mathematician avenues to power hitherto beyond the reach of scientific ambition; while chemistry has become mistress of the arts of life, since *quantitative* analysis has been introduced in her researches.

194. Arithmetic is the necessary link between all mathematics and their purely practical applications.

For, in all purely practical applications, there must be measurement, and reference to a unit. And this can only be applied to discontinuous quantity. Hence, those who define mathematics as the science of measurement make it wholly dependent on arithmetic, (192), on which it depends, however, only for its application to practical questions, or to discontinuous quantities.

195. In its scientific interest alone, arithmetic offers an unbounded field.

For the combinations and properties of numbers, though not strictly infinite, must be practically so.

The character of this treatise as a mere introduction forbids us to enter into the tempting field of the theory of numbers. The learner is referred, for his future studies in arithmetic, to Leslie's "Philosophy of Arithmetic," Barlow's "Theory of Numbers," Gauss's "Disquisitiones Arithmeticæ," Legendre's "Theorie des Nombres," Lenhart's papers in Professor Gill's "Mathematical Miscellany," and articles in different Encyclopædias.

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'We value the book, because it aims not merely to give demonstrations — to discipline the mind, as it is called, by bare logic, without practical facts, but, on the contrary, aims directly at the knowledge of facts and things, and the useful and entertaining employment of them. This is the way to discipline the mind — to *feed* it, with good food.'—*Portland Advertiser.*

The editor of the North American Review, says: 'The tendency of the youthful mind to observe and study forms is developed by this book. * * Geometry is taken up as a science, not only severe, but comprehensive, when the pupil is too young to appreciate its scope and importance, and he must commence his hard task with definitions. It is a high merit of this "Introduction," that it teaches these definitions in a familiar way, long before they are required as the preliminary steps to abstract study. It also supplies such a rudimentary knowledge of the principles of Geometry, and their application, as lessens, if it does not remove, the difficulties attending the first efforts in theoretical reasoning. The object of the "Introduction" which it is eminently suited to attain, is to "help children to arrange what nature teaches."'

'It is decidedly the best work of the kind we have ever seen, and, more than any other with which we are acquainted, will win the interest of the beginner to this science. Geometry is greatly neglected as a branch of education, owing, in some measure, doubtless, to the unintelligibility of text-books to the young. The "Introduction" is very clear in its explanations, and its exercises combine pleasure with instruction.'—*Salem Obs.*

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"There is one subject connected with sound, [illegible] which we think Professor Peirce has [illegible] the organs of the human voice. There have been very [illegible] in regard to the peculiar [illegible] of each part of this complicated structure. In Mr. Peirce's book it is shown how they might be [illegible]."—*North American Review.*